

Finite temperature phase transition of a single scalar field on a fuzzy sphere

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We study finite temperature phase transition of neutral scalar field on a fuzzy sphere using Monte Carlo simulations. We work with the zero mode in the temporal directions, while the effects of the higher modes are taken care by the temperature dependence of r . In the numerical calculations we use “pseudo-heatbath” method which reduces the auto-correlation considerably. Our results agree with the conventional calculations. We report some new results which show the presence of meta-stable states and also suggest that for suitable choice of parameters the symmetry breaking transition is of first order.

PACS numbers: 12.60.Rc; 12.10.-g; 14.80.Hv; 11.25.Wx; 11.10.Hi

I. INTRODUCTION

QFT's on non-commutative spaces have been studied from various perspectives recently [1, 2, 3, 4, 5, 6, 7, 8]. Most frequently studied NC space is the well known Groenwald Moyal space [9] R_θ^{2d} and various issues like, renormalisation, causality, solitons, statistics have been analysed in the literature [10, 11, 12, 13]. The conventional quantisation of fields on these spaces have led to an interesting behavior known as IR/UV mixing. The phase structure of fields on such a space reveals a new phase known as strip phase [14]. Alternative quantisation which preserves a twisted Poincare symmetry in these theories avoids such a difficulty [15, 16].

On the other hand the fields on fuzzy spaces like fuzzy spheres, fuzzy CP_n etc are explicitly finite and do not have the IR/UV mixing [17, 18]. But there is an anomaly in the finite case which reveals itself as generating the IR/UV mixing. There is lot of confusion about taking the limit of continuum in these models and it has been pointed out various possibilities do exist [17, 18, 19, 20].

The QFT on fuzzy sphere is a matrix model and easily amenable to simulations and numerical studies [21, 22, 23]. We study a real scalar field on the fuzzy sphere using Monte Carlo simulations. The earlier studies involved metropolis algorithm to ensure the randomness of fluctuations but the autocorrelations are reduced using over-relaxations [23]. But we will use another technique extensively used in the study of Higgs model - known as pseudo-heatbath [24]. Using this algorithm we have been able to reproduce previous results from different studies. Apart from this we are able to characterise the order of the transitions between the different phases. In particular we find the transition between order \leftrightarrow non-uniform transition is of first order which is mostly due to the presence of many meta-stable states at low temperatures in the model. Also we find new results for the structure of the phase diagram as well as for scaling of the location of the triple points in the continuum limit.

The paper is organised as follows: the sec(2) introduces the QFT on fuzzy spheres; sec(3) discusses the pseudo-heatbath technique and brings out its salient features. Sec(4) reproduces the known

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results and discusses our results on the nature of phases and the transitions. In sec(5) we present our conclusions.

II. THE MODEL

We use the following action for the massive, neutral, scalar field on the fuzzy sphere of radius R [19, 21, 23],

$$S(\Phi) = \frac{4\pi}{N} \text{Tr} [\Phi [L_i, [L_i, \Phi]] + R^2 (r\Phi^2 + \lambda\Phi^4)] . \quad (1)$$

Here $\Phi \in \text{Mat}_N$ is a $N \times N$ hermitian matrix. The first term in the action is (kinetic) coming from the variation of the Φ field on the fuzzy sphere. The quartic term represents the self interaction of the Φ field. For the thermal behavior of the Φ field one needs to study the system in $2+1d$. However one can consider the above action as the dimensionally reduced version of a $2+1$ dimensional action with the effects of temperature going into the temperature dependence of r . Finite temperature behavior of the Φ field with fluctuations included is then studied at different r . In the mean field approximation the expectation value of Φ field which minimises the action is given by,

$$\langle \Phi \rangle = \pm \phi \mathbb{1}, \quad \phi = \sqrt{-\frac{r}{2\lambda}} \quad (2)$$

$\phi \geq 0$ for negative values of r and zero for $r \geq 0$. The system is in ordered phase for $r < 0$ and in the disorder phase for $r \geq 0$. So at the mean-field level there are only two phases. $\langle \Phi \rangle$ decreases continuously to zero in the limit $r \rightarrow 0$. At $r = 0$ the system undergoes a second order phase transition with mean field critical exponents, $\beta = 1/2, \alpha = 0$ etc.. In the disorder phase $\langle \Phi \rangle = 0$ so the Z_2 symmetry of the model is restored. Even though the above form of $\langle \Phi \rangle$ minimises the action there are additional local minima or meta-stable states. For these states, form of $\langle \Phi \rangle$ is non-identity in general. Some of these states will have same potential as the ground state. In the mean-field approach these states do not play any role in the phase transition. However they become important when fluctuations beyond mean-field are considered.

Hence the next step in the calculations is to consider effect of thermal fluctuations beyond mean field. It is important then to ask if the results of the mean field analysis survive. One expects that the fluctuations will destroy a non-zero $\langle \Phi \rangle$ even for r less than the mean-field critical value which is zero. Further more the fluctuations may lead to new phases and different types of phase transitions. These are some of the issues of intense numerical investigations recently [19, 21, 23]. So far the results show the appearance of a new phase called non-uniform ordered phase. These studies are mostly done using a standard Monte Carlo simulations with metropolis algorithm.

In the present work we study the fluctuations in the above model using a different numerical technique known as “pseudo-heatbath” method. Like the previous studies we also observe the non-uniform phase. However our results seem to indicate that there are phase structure within the non-uniform phase. These phases can be probed using different operators/order parameters. As a consequence there will be multiple triple points in the $\lambda - r$ plane. In the following we describe the “pseudo-heatbath” method. Subsequently we present and discuss our results next section.

III. NUMERICAL TECHNIQUE

Effects of the fluctuations beyond mean field are computed from the partition function, which in the path integral approach is given by,

$$\mathcal{Z} \propto \int D\Phi e^{-S(\Phi)}. \quad (3)$$

The standard numerical methods adopted for this integration are Monte Carlo simulations. In the Monte Carlo algorithms, one generates an “almost” random sequence of Φ matrices by successively updating elements of Φ taking into account the measure and the exponential in the integral above. This sequence of Φ is then used as an ensemble for calculating averages of various observables. For example, thermal average of Φ is given by,

$$\langle \Phi \rangle = \frac{1}{N_m} \sum_{m=1}^{N_m} \Phi_m, \quad (4)$$

here, Φ_m is the m th element of the ensemble. Usually there are different ways to generate the ensemble. Previous studies of this model have considered the metropolis algorithm [21, 23]. In the metropolis updating usually there is a substantial correlation between Φ 's in the sequence. For a good ensemble the auto correlation between the configurations in the sequence must be really small. Though this auto correlation can be reduced by using some over relaxation programme [23]. The auto-correlation is greatly reduced, however, when “heatbath/pseudo-heatbath” type of algorithms are used [24]. This method is very much common in the non-perturbative study of Φ^4 theories in conventional lattice simulations. It gives better sampling and is efficient at least for smaller λ values. This is why we make use of “pseudo-heatbath” technique. In the following we explain the algorithm in greater detail.

In the “pseudo-heatbath” algorithm, given a Φ we update the elements of this matrix one at a time. Advantages of updating matrix elements were demonstrated in Ref. [25]. Keeping in mind that Φ is hermitian we update Φ_{ij} and Φ_{ji} simultaneously. We update Φ_{ij} using the probability distribution,

$$P(\Phi_{ij}) = e^{-S(\Phi_{ij})} \\ \text{where} \quad S(\Phi_{ij}) = \alpha(\Phi_{ij} - A)^2 + \lambda B(\Phi_{ij} - C)^4, \quad (5)$$

A, B, C may depend on the elements of Φ (other than Φ_{ij}). α is a parameter chosen so as to maximise the efficiency of updating. In the first step a random number is generated using the distribution,

$$e^{-\alpha(\Phi_{ij} - A)^2}. \quad (6)$$

In the second step the newly generated random number is accepted or rejected using the second term of $S(\Phi_{ij})$. In our calculations we get for some choice of the parameters, in particular small λ acceptance rate up to 95%. Over relaxation can also be easily incorporated into this algorithm. In the over relaxation process we flip the element $\phi = \Phi_{ij}$ in the following way,

$$\phi' = A - 2\phi \quad (7)$$

then accept it with the probability $\exp(-\delta S)$. δS is the change in action due to flipping. For small λ this amounts to changing Φ by large amount with only a small change in the total action. Even without using the over relaxation method we get very small auto correlation. In the following we present and discuss our numerical results.

IV. NUMERICAL RESULTS AND DISCUSSION

To study the phase diagram and transitions we make measurements of various observables such as

$$Tr(\Phi), \quad Tr(\Phi^2), \quad Tr(S) \quad (8)$$

at various values rR^2 for different choices of $(N, \lambda R^2)$. In order to check our algorithm we considered some of the parameters used in previous calculations [19, 21, 23], and found that our results match reasonably well with previous results. The results also agreed with mean-field away from the transition point. In Fig. 1(a) we show Monte Carlo history of $Tr(\Phi)$ for $N = 2$, $R = 1.0$, $\lambda = 0.636620$ and $r = -1.530502$ as used in [23]. Without using the over relaxation we get the quality of data similar to that of [23] shown in Fig. 1(b). The average values agree but we observe larger fluctuations (Fig. 1a).

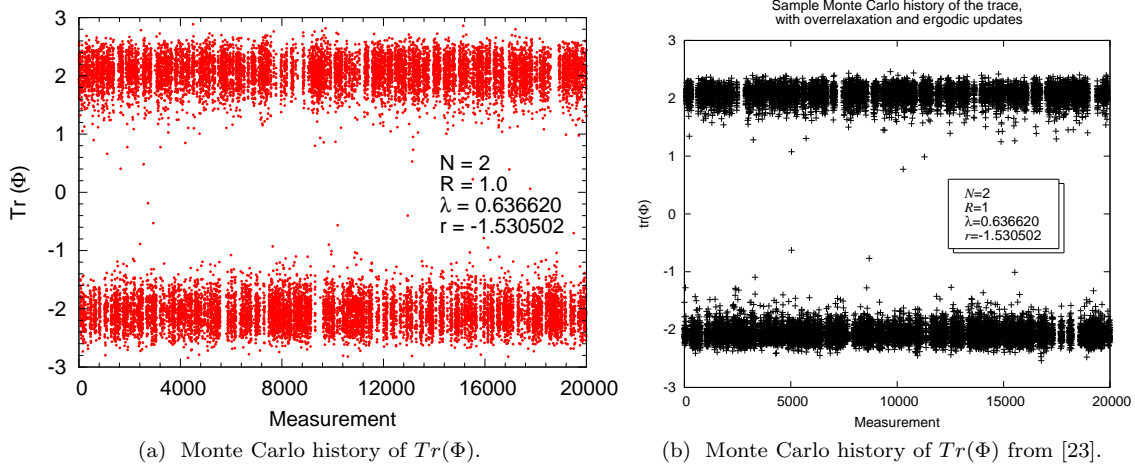


FIG. 1: Comparison between pseudo-heatbath method and Metropolis method

Order \leftrightarrow non-uniform transition

Having reproduced some of the previous results we considered various values of the parameters $(N, \lambda R^2)$ to study the phase diagram. For large values of λR^2 we found multiple transitions [19]. For low temperatures, or $r \ll 0$, the average value of Φ is essentially identity matrix indicating the ordered phase. For larger values of r the average of $Tr(\Phi)$ vanishes while the average of some elements Φ_{ij} is non-zero. Such a form of Φ average indicates a non-uniform phase which breaks spatial rotation spontaneously. Since the average of $Tr(\Phi)$ is non-zero in the order phase and zero in the non-uniform phase one can use it as an order parameter for the order \leftrightarrow non-uniform transition.

As for the order of the non-uniform \leftrightarrow order transition we find it is strong first order for larger λR^2 . This can be seen from the hysteresis effects of $Tr(\Phi)$. In Fig. 2(a) we show the hysteresis loop of $Tr(\Phi)$ for the set of parameters $N = 25$ and $\lambda = 0.8$. The value of r corresponding to the middle of the hysteresis loop is take to be the critical(transition) value $r = r_1$ for this transition. By doing simulations for different values of λ we find that the strength of this order \leftrightarrow non-uniform transition varies. For smaller λ the transition becomes weaker. For example, for $\lambda = 0.4$ the hysteresis loop was not prominent.

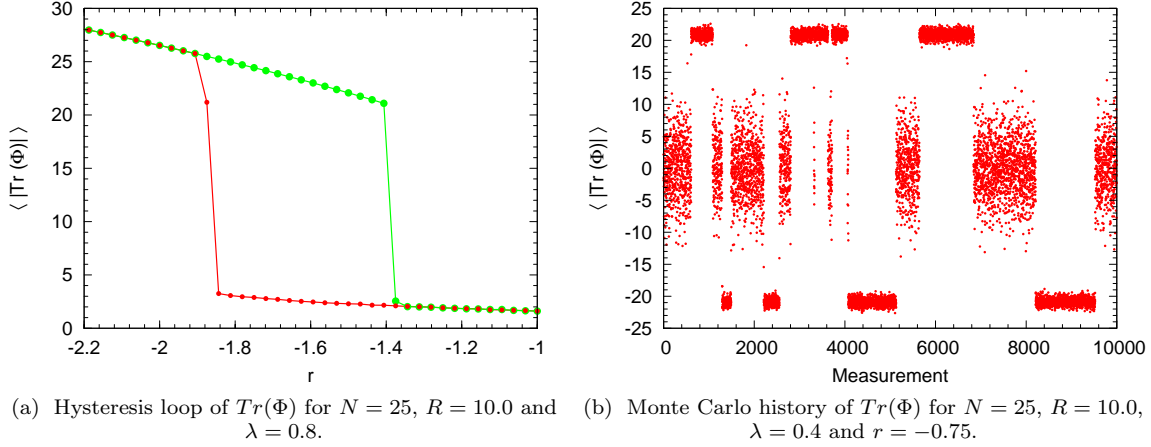


FIG. 2: First-order Phase Transition

The history of measurement of $Tr(\Phi)$ and its distribution is shown in Fig. 2b for r close to the corresponding critical value r_1 . One clearly sees three degenerate ground states here. Out of these two are connected by Z_2 symmetry and the 3rd has $Tr(\Phi)$ peaked around zero and represents a non-uniform ordered phase. This distribution indicates that the order and non-uniform phases do coexist suggesting first order nature of the transition between these two phases. Even though the transition is first order, it is weaker compared to the previous example.

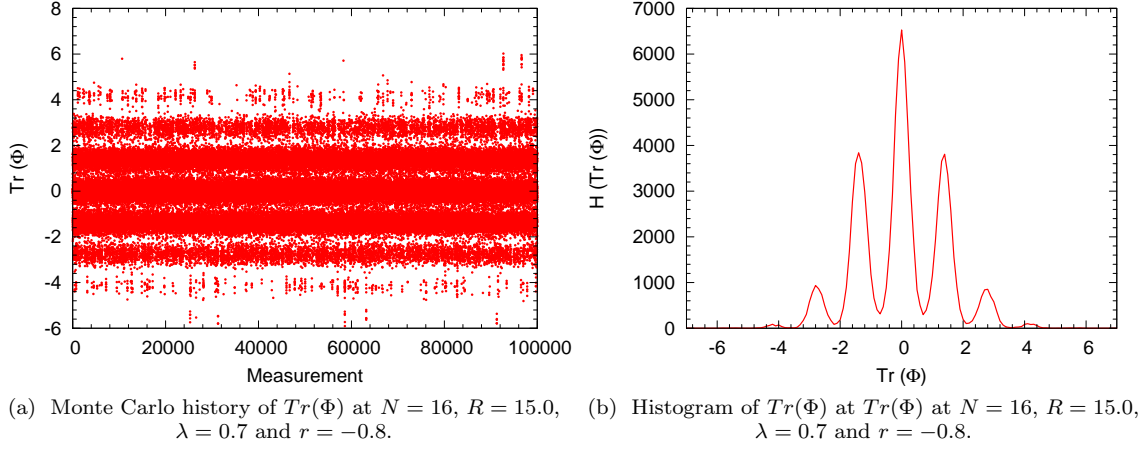
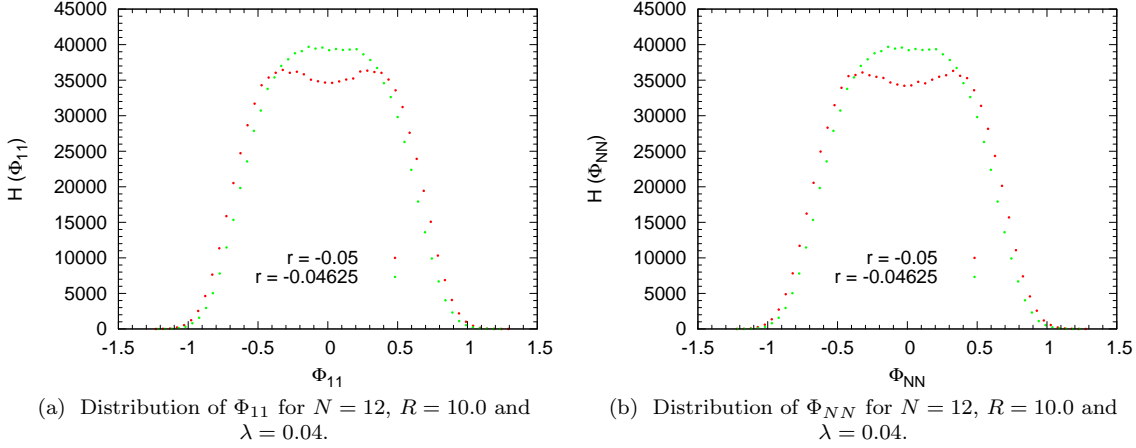
Non-uniform \leftrightarrow disorder transition

In the non-uniform phase $Tr(\Phi)$ keeps fluctuating around zero. One can clearly see measured values of $Tr(\Phi)$ form bands symmetrically situated around zero as shown in figure Fig. 3(a). The band structure are not seen in the Monte Carlo history of $Tr(\Phi^2)$ and $S(\Phi)$. This band structure of $Tr(\Phi)$ we saw mostly in the case when there were many meta-stable states before transition in the ordered phase, i.e for larger N . In Fig. 3(b) we show the histogram of $Tr(\Phi)$ which clearly shows a peak close to zero. This implies the state with lowest $Tr(\Phi)$ is the ground state of the system and other bands are meta-stable states. The meta-stable bands tend towards zero as we increase r further. At the same time some bands disappear and/or others merge with the middle one.

In the basis we choose to work with even though $Tr(\Phi)$ fluctuates around zero both Φ_{11} and Φ_{NN} fluctuate around non-zero Z_2 symmetric values. So the symmetry of Φ is not restored yet. When we increase r the non-zero values around which the first and last diagonal of Φ fluctuate approach smoothly to zero. Beyond certain value of $r = r_2$ all elements of Φ fluctuate around zero restoring the Z_2 symmetry. In Fig. 4 we show the distribution of these elements of Φ both below and above r_2 . Given this behavior of Φ_{11} and Φ_{NN} one can consider any of them as the order parameter for the non-uniform \leftrightarrow disorder transition. This also implies that the higher spherical harmonics are becoming important for this transition in our basis [19]. When r_2 is approached from below the peaks of the distribution of Φ_{11} , Φ_{NN} smoothly approach zero indicating only one ground state at any particular value of r around the transition point. So this transition is a continuous transition.

Phase diagram and triple points

When we analysed the data for fluctuations of $Tr(\Phi^2)$ we found these peaked at a certain value

FIG. 3: Monte Carlo history and histogram for $r_1 < r < r_2$ FIG. 4: Non-uniform \leftrightarrow disorder transition

of r within range $r_1 < r < r_2$ (Fig. 5b). This suggests finer structure or phases in the non-uniform phase. The peak in $Tr(\Phi^2)$ then corresponds to the transition between these phases.

We did not see any dramatic change in the variables such as $Tr(\Phi)$ or the diagonal elements of Φ . The finiteness of the peak implies that the transition is a continuous transition. For larger λ the transition point was far from both r_1 and r_2 . We anticipate that the non-uniform phase has more structure than what we see from the behavior of $Tr(\Phi^2)$. This finer structure could be explored by appropriate operators such as multi-trace operators. Note that this fine structure becomes more prominent for larger N , which implies that it will survive in the continuum non-commutative limit.

As r is increased from some large negative value the system explores all these phases for large λ . For smaller λ some of these phases will not appear when r is varied. This leads to presence of triple points in the $\lambda - r$ plane. For some small λ there is transition directly between order \leftrightarrow disorder

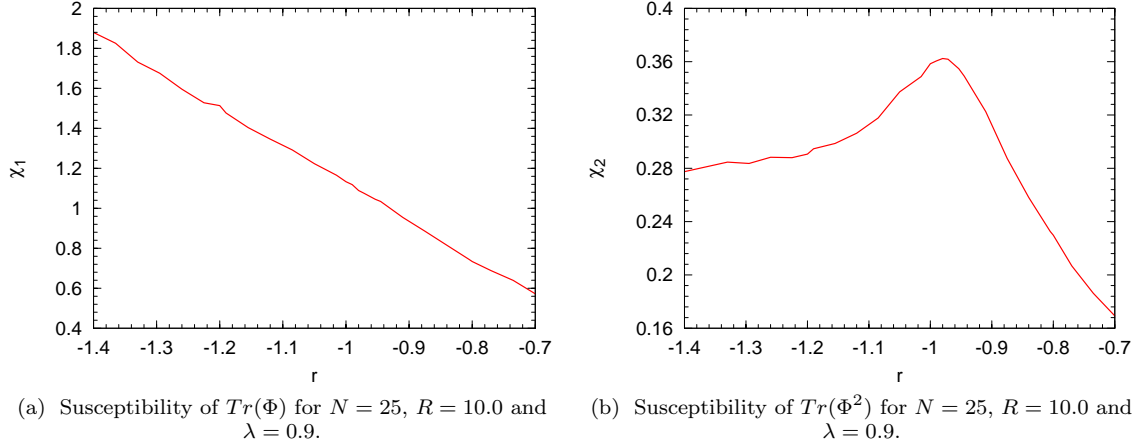


FIG. 5: Transition between non-uniform phases

phases, i.e $|r_1 - r_2|$ vanishes. This leads the triple point which has the lowest value of λR^2 . In Fig. 6a we show the phase diagram for $N = 25$ in the λR^2 vs $r R^2$ plane. This is the triple point studied in previous works. In these studies the triple point was obtained by using numerical results for the order-non-uniform transition and the analytic results which takes into account only the potential term [19]. In our case both transitions lines are from our simulations.

Conventional lattice regularization of the model does not show any evidence of non-uniform ordered phase. So it is imperative to study what happens to the non-uniform ordered phase in the continuum limit. If the non-uniform phase survives this limit then only it can be physically relevant. We have studied the N dependence of the triple point. Since it's not practical to do simulations for very large N , one must study scaling to find out the limiting position of the triple point for larger N values. In the Fig. 6b we show value of $Y_{tri} = \lambda R^2$, $X_{tri} = r R^2$ corresponding to the triple point for different N . The values of N considered for our simulations are $N = 4, 8, 12, 16, 25$. When N was increased the triple point moved away from the origin. Our results suggest that the triple point scales with (N^μ, N^ν) with $\mu \simeq 1.0 \simeq \nu$. We did not observe any universal scaling of the phase boundaries in the phase diagram for different N .

For smaller λR^2 there is only one transition, the order \leftrightarrow disorder transition. For larger N the distribution of the observables such as $Tr(\Phi)$, Φ_{11} , Φ_{NN} , close to the critical point, show a plateau around zero with highly non-gaussian features. This can be seen in Fig. 7 where we have shown the histogram of $Tr(\Phi)$, Φ_{11} around the critical temperature. The parameters considered here are $\lambda = 0.4$, $R^2 = 10^2$, $N = 12$. We take the plateau structure around zero as an indication of a transition which is stronger than second order transition. However it does not rule out the possibility of second order phase transition for smaller values of λ and N .

Meta-stable states

We also studied cases with very large values of λR^2 . When r is large negative we find different average values for $Tr(\Phi)$ for different initial choices of Φ . These different values correspond to local and global minima of the effective action. The barrier between these states inhibits the transition amongst them. The number of these states which we observed grow with N . This can be more or less seen from the analysis of the action itself as fluctuations are not much important for small r . The state with highest $Tr(\Phi)$ found to satisfy $\langle \Phi \rangle \propto \mathbb{1}$ and has the lowest value for the action,

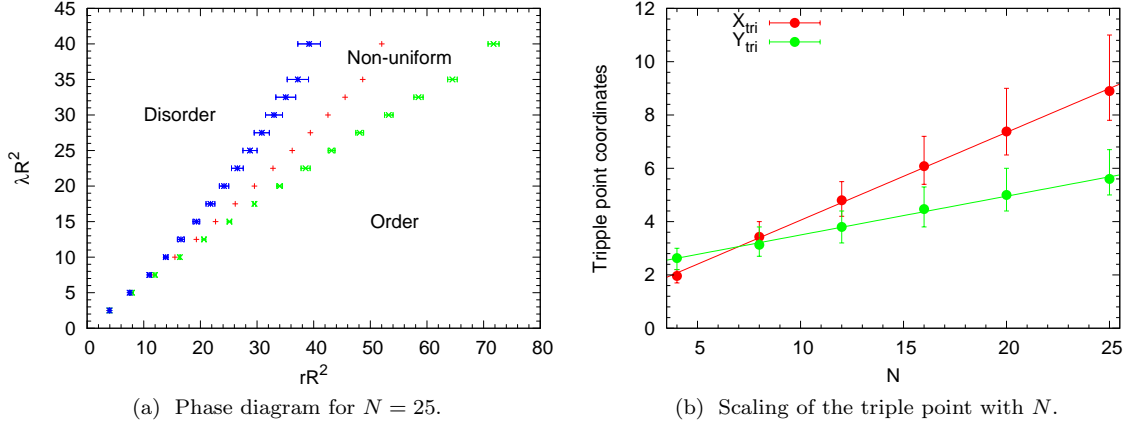


FIG. 6: Phase diagram and scaling of the triple point

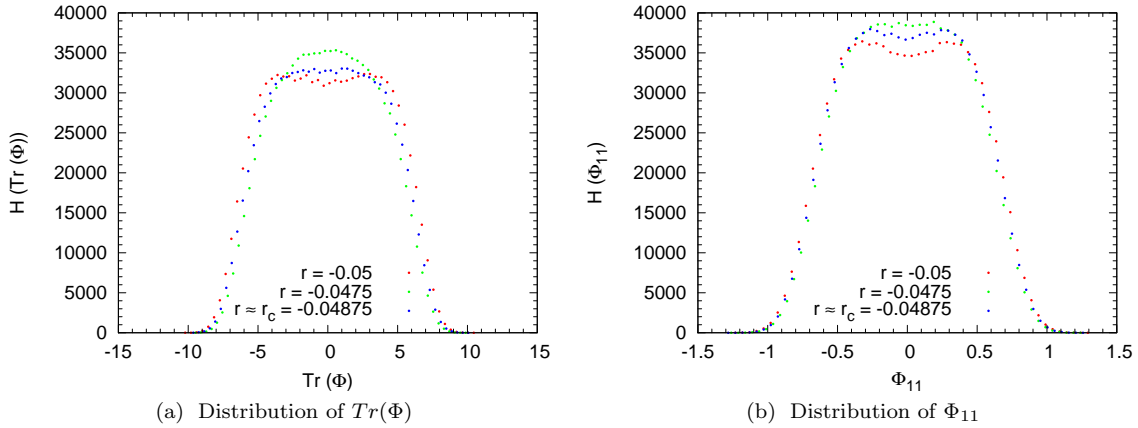
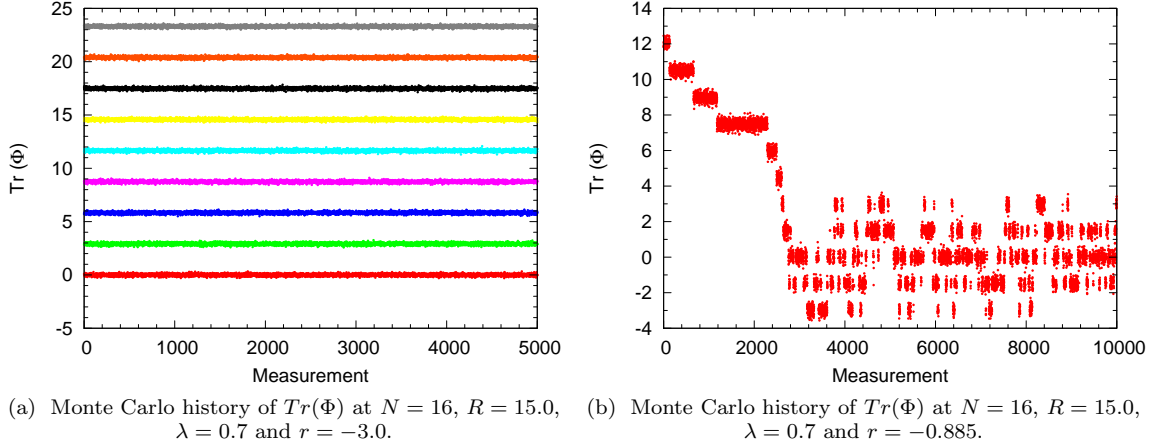


FIG. 7:

hence is the ground state. So we conclude that the state with largest $\text{Tr}(\Phi)$ is the global minimum of the system. Other states, which are basically the non-uniform phases, are meta-stable. We think that the meta-stability increases with decrease in the average of $\text{Tr}(\Phi)$. In Fig. 8(a) we show a brief Monte Carlo history of Φ , after thermalization, for different initial Φ 's. For higher r the bands persist but move slowly towards zero. After certain value of critical r most of these states are observed only for sometime in the Monte Carlo history and then the values of $\text{Tr}(\Phi)$ jumped to zero as seen in Fig. 8(b).

V. CONCLUSIONS

We have developed a “pseudo-heatbath” algorithm to study the finite temperature phase transitions of Φ^4 theory on a fuzzy sphere. The results from Monte Carlo simulations clearly show

FIG. 8: Monte Carlo history for $r < r_1$ and for $r \sim r_1$

finite temperature transitions. For some range of λR^2 , in particular, for large values one clearly sees stable non-uniform phases for some intermediate temperature, intermediate values of r . The various phases are characterised by different properties of Φ . In the ordered phase this behaves like a identity matrix. All non-uniform phases have zero $Tr(\Phi)$. Their existence is confirmed by the peak in the fluctuation of $Tr(\Phi^2)$. $Tr(\Phi)$ serves as an order parameter for the order \leftrightarrow non-uniform transition while Φ_{11} and Φ_{NN} describe the non-uniform \leftrightarrow disorder transition.

The order-non-uniform transition is found to be first order. This transition was found to be strong first order for larger values of λR^2 . We conjecture that the first order nature of the transition has to do with the presence of meta-stable states discussed above. In fact the state with $Tr(\Phi) = 0$ is meta-stable for small temperatures when $\Phi \propto \mathbb{1}$ is the absolute ground state. Fluctuations can only stabilise if it is a stable configuration at higher temperature, so there is always a barrier with the ordered phase, leading to first order transition. For smaller values of λR^2 the barrier between the stable and meta-stable phases is not high so thermal fluctuations make Φ hop between the different states. In this case we rather study the distribution of $Tr(\Phi)$ to infer the transition value of r for the order \leftrightarrow non-uniform transition. From the distribution we find that ground state is discontinuously changing. Moreover the distribution of $Tr(\Phi)$ is very non-gaussian, rather flat near zero, suggesting that there are degenerate states characterised by zero and non-zero values $Tr(\Phi)$. So from our results the transition of ordered phase to other phases is always first order for the parameter space we have explored. All other transition in model appeared to be continuous transitions.

The results from previous studies have shown that by doing simple scaling phase boundaries for different N coincide [19]. This expected scaling with N does not occur up to the value of N we have studied, though it is the largest so far. We studied the behavior of the triple point for large N and it scales approximately linearly with N . However our results seem to agree with the previous studies in that the non-uniform phase survives the continuum limit. In our analysis we considered primarily $Tr(\Phi)$, $Tr(\Phi^2)$, Φ_{11} etc.. However analysis of the full matrix may result in better understanding of the phase structure, such as variants of the non-uniform ordered phases.

VI. ACKNOWLEDGEMENT

We thank D. O'Connor, M. Panero, X. Martin, W. Bietenholz for critical comments on the draft of this paper. All the calculations have been carried out using the computer facilities at IMSc, Chennai.

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- [1] M.R. Douglas and N.A. Nekrasov, *Rev.Mod.Phys.* **73**, 977 (2001).
 - [2] R.J. Szabo, *Class.Quant.Grav.* **23**, R199 (2006).
 - [3] D. Karabali, V.P. Nair and S. Randjbar-Daemi, *arXiv:hep-th/0407007*.
 - [4] A. Connes, M.R. Douglas and A.S. Schwarz, *JHEP* **9802**, 003 (1998).
 - [5] R.J. Szabo, *Phys.Rept.* **378**, 207 (2003).
 - [6] A.P. Balachandran, S. Kurkcuoglu and S. Vaidya, *arXiv:hep-th/0511114*.
 - [7] A.P. Balachandran, T.R. Govindarajan, C. Molina and P. Teotonio-Sobrinho, *JHEP* **0410**, 072 (2004).
 - [8] A.P. Balachandran, T.R. Govindarajan and B. Ydri, *Mod.Phys.Lett.* **A15**, 1279 (2000).
 - [9] J.E. Moyal, *Proc.Cambridge Phil.Soc.* **45**, 99 (1949).
 - [10] S. Minwalla, M.V. Raamsdonk and N. Seiberg, *JHEP* **0002**, 020 (2000).
 - [11] R. Gopakumar, S. Minwalla and A. Strominger, *JHEP* **0005**, 020 (2000).
 - [12] M. Chaichian, K. Nishijima and A. Tureanu, *Phys.Lett.* **B568**, 146 (2003); M. Chaichian, P. Kulish, K. Nishijima and A. Tureanu, *Phys.Lett.* **B604**, 98 (2004).
 - [13] O.W. Greenberg, *Phys.Rev.* **D73**, 045014 (2006).
 - [14] S.S. Gubser and S.L. Sondhi, *Nucl.Phys.* **B605**, 395 (2001).
 - [15] A.P. Balachandran, G. Mangano, A. Pinzul and S. Vaidya, *Int.J.Mod.Phys.* **A21**, 3111 (2006).
 - [16] A.P. Balachandran, T.R. Govindarajan, G. Mangano, A. Pinzul, B. A. Qureshi and S. Vaidya, *Phys.Rev.* **D75**, 045009 (2007).
 - [17] S. Vaidya and B. Ydri, *Nucl.Phys.* **B671**, 401 (2003); *arXiv:hep-th/0209131*.
 - [18] Chong-Sun Chu, J. Madore and H. Steinacker, *JHEP* **0108**, 38 (2001).
 - [19] J. Medina, W. Bietenholz, F. Hofheinz and D. O'Connor, *PoS LAT2005*, 263 (2005).
 - [20] H. Steinacker, *JHEP* **0503**, 075 (2005).
 - [21] X. Martin, *JHEP* **0404**, 077 (2004).
 - [22] W. Bietenholz, F. Hofheinz and J. Nishimura, *Acta Phys.Polon.* **B34**, 4711 (2003).
 - [23] M. Panero, *JHEP* **0705**, 082 (2007); *SIGMA* **2**, 081 (2006).
 - [24] B. Bunk, *Nucl.Phys.Proc.Suppl.* **42**, 566, (1995).
 - [25] W. Bietenholz, F. Hofheinz and J. Nishimura, *JHEP* **0406**, 042 (2004).